

CIS 5560

Cryptography Lecture 15

Course website:

pratyushmishra.com/classes/cis-5560-s25

Recap of Last Lecture(s)

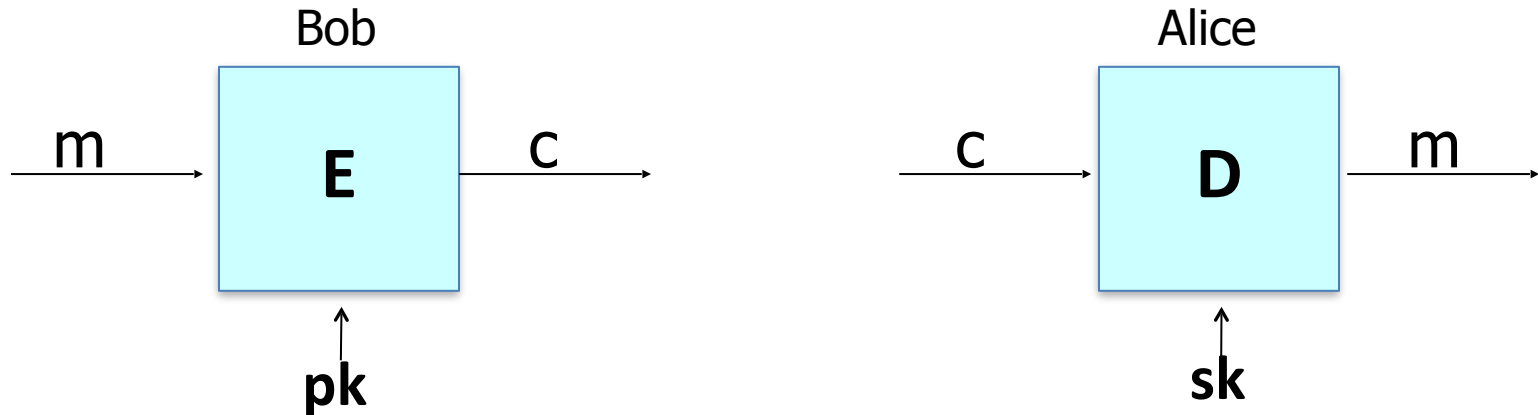
- Number Theory refresher
 - Arithmetic modulo primes
 - Fermat's Little Theorem
 - Cyclic groups
 - Discrete Logarithms
- Key Exchange
 - Merkle puzzles
 - Diffie—Hellman
 - Computational Diffie—Hellman Problem

Today's Lecture

- Public Key Encryption
 - El Gamal Encryption
 - Computational Diffie—Hellman Problem
 - RSA Encryption
 - Arithmetic modulo composites
 - Factoring

Public key encryption

Alice: generates (PK, SK) and gives PK to Bob



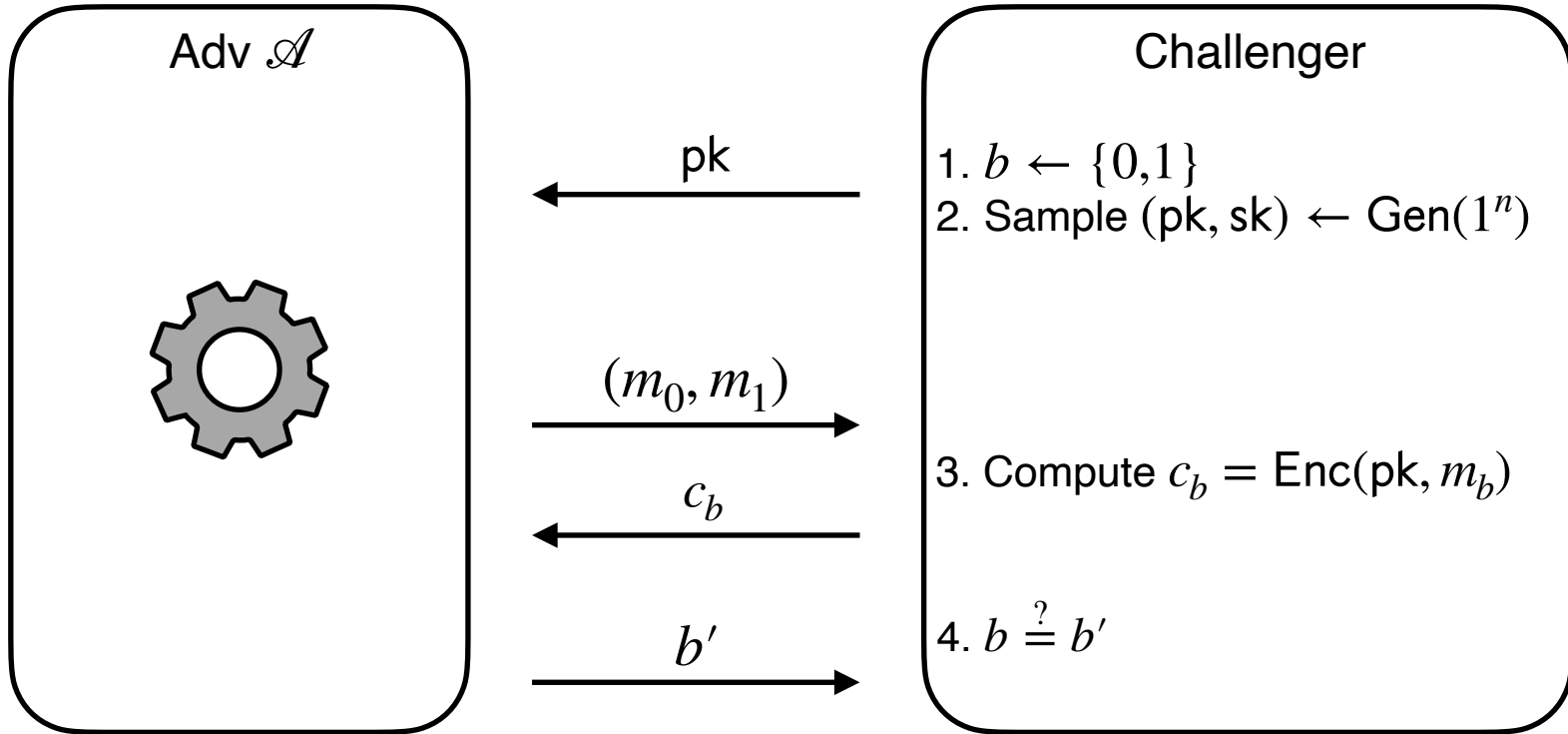
Public key encryption

Def: a public-key encryption system is a triple of algs. (G, E, D)

- $\text{Gen}()$: randomized alg. outputs a key pair (pk, sk)
- $\text{Enc}(pk, m)$: randomized alg. that takes $m \in \mathcal{M}$ and outputs $c \in \mathcal{C}$
- $\text{Dec}(sk, c)$: deterministic alg. that takes $c \in \mathcal{C}$ and outputs $m \in \mathcal{M} \cup \{ \perp \}$

Correctness: $\forall (pk, sk) \text{ output by } \text{Gen}(), \forall m \in \mathcal{M}, \text{Dec}(sk, \text{Enc}(pk, m)) = m$

Security: IND-CPA for PKE



$$\Pr[b = b'] = 1/2 + \text{negl}(n)$$

Security: IND-CPA for PKE

For all PPT adversaries \mathcal{A} , the following holds:

$$\Pr \left[b = \mathcal{A}(\text{Enc}(\text{pk}, m_b)) \mid \begin{array}{l} (\text{pk}, \text{sk}) \leftarrow \text{Gen}(1^n) \\ \text{Sample } b \leftarrow \{0,1\} \\ (m_0, m_1) \leftarrow \mathcal{A}(\text{pk}) \end{array} \right] \leq \text{negl}(n)$$

How does it relate to symmetric-key IND-CPA?

Recall: for symmetric ciphers we had two security notions:

- One-time security and many-time security (CPA)
- We showed that one-time security does not imply many-time security

For public key encryption:

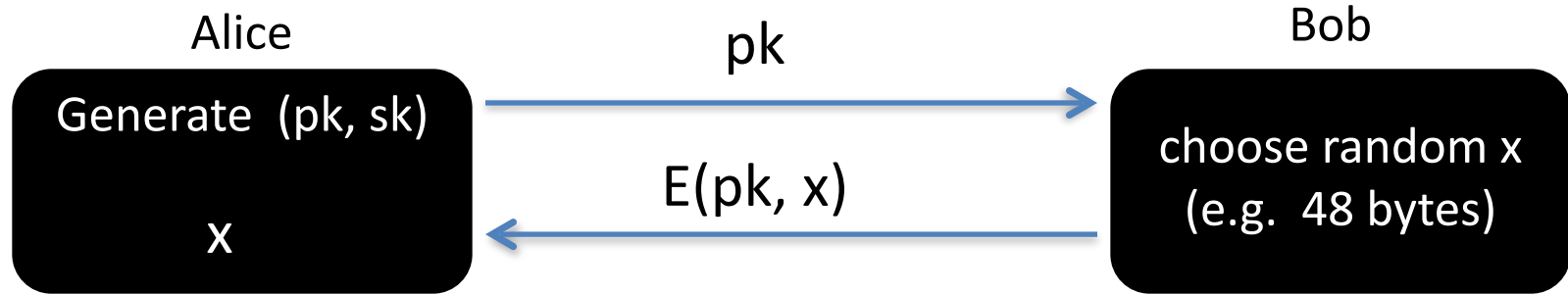
- One-time security \Rightarrow many-time security (CPA)

(follows from the fact that attacker can encrypt by himself)

- Public key encryption **must** be randomized

Applications

Session setup (for now, only eavesdropping security)



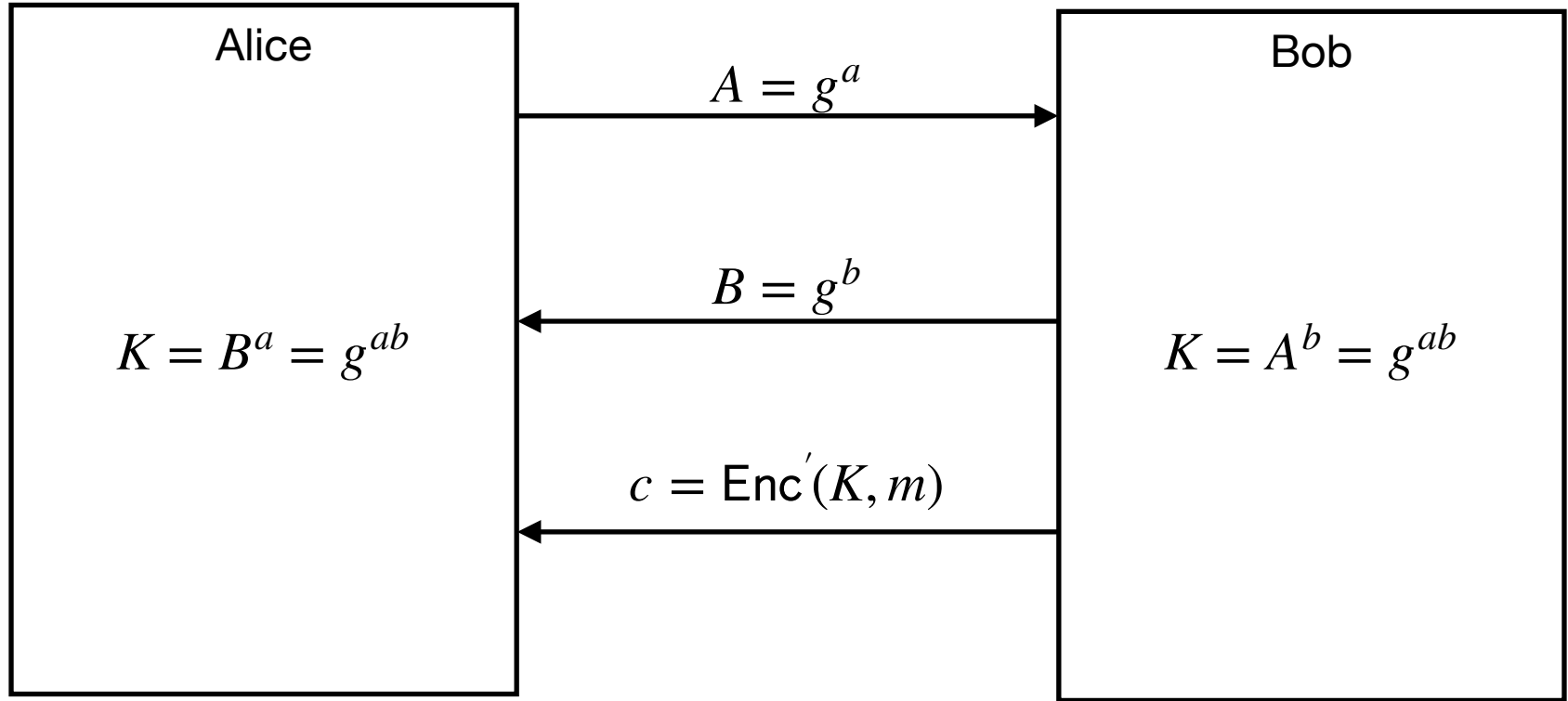
Non-interactive applications: (e.g. Email)

- Bob sends email to Alice encrypted using pk_{alice}
- Note: Bob needs pk_{alice} (public key management)

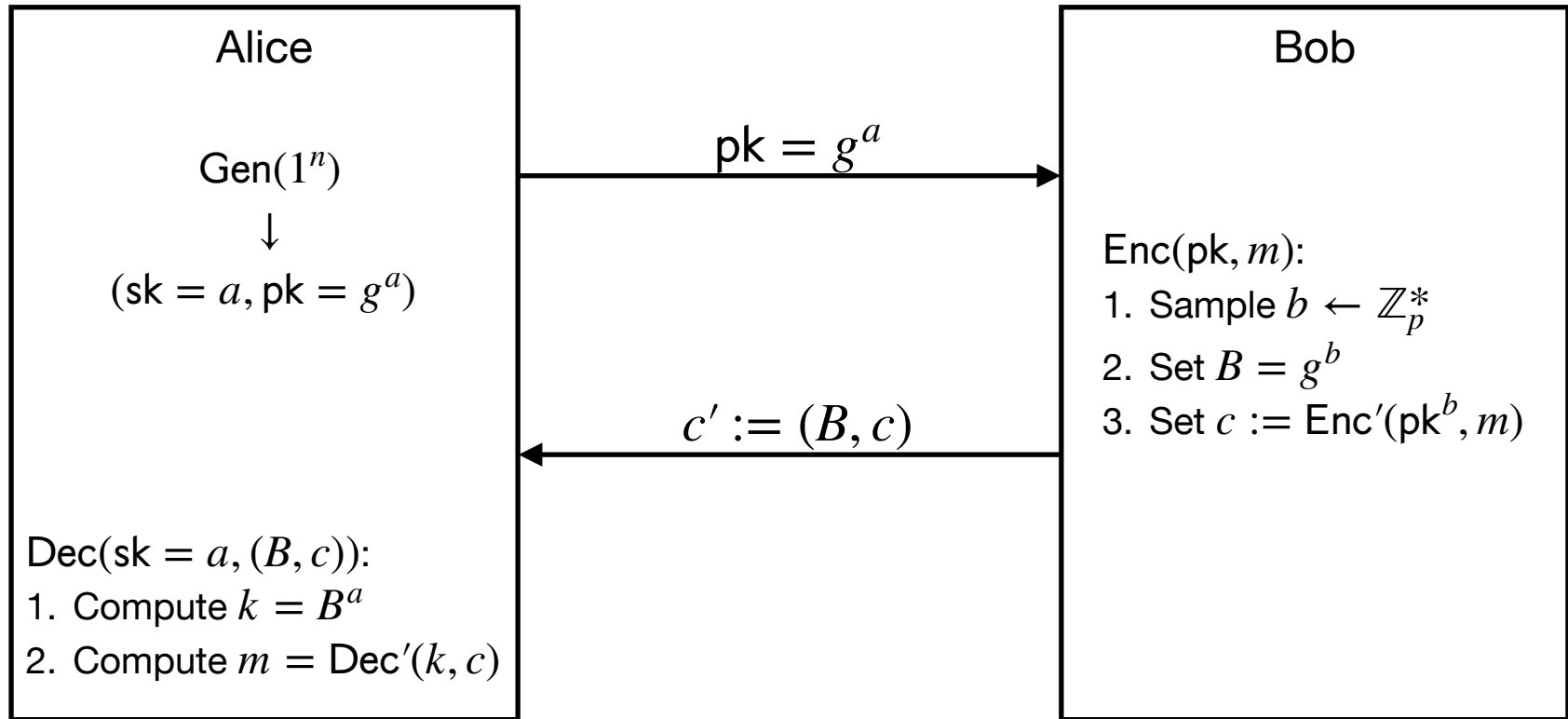
Constructions of PKE: Elgamal Encryption

Review of cyclic groups (On board)

Recall: DH Key Exchange



Convert DH \rightarrow PKE



The Elgamal system (an abstract view)

- \mathbb{G} : finite cyclic group of prime order p with generator g
- $(\text{Enc}', \text{Dec}')$: symmetric-key encryption with keyspace $\mathcal{K} = \mathbb{G}$

Gen(1^n):

1. Sample $a \leftarrow \mathbb{Z}_p^*$
2. Output $(\text{sk} = a, \text{pk} = g^a)$

Enc(pk, m):

1. Sample $b \leftarrow \mathbb{Z}_p^*$
2. Set $B = g^b$
3. Set $c := \text{Enc}'(\text{pk}^b, m)$
4. Output $c' = (B, c)$

Dec(sk = a , (B, c)):

1. Compute $k = B^a$
2. Output $m = \text{Dec}'(k, c)$

What choice of $(\text{Enc}', \text{Dec}')$?

How to prove security?

Q1: Choice of $(\text{Enc}', \text{Dec}')$: OTP?

- \mathbb{G} : finite cyclic group of prime order p with generator g
- Key idea: One-Time Pad works not just with $\{0,1\}^n$ and XOR, but with *any group*
 - $\text{Gen}'(1^n)$: Sample $r \leftarrow \mathbb{Z}_p$, and output g^r
 - $\text{Enc}'(k = g^r, m \in \mathbb{G})$: Output $c = k \cdot m \in \mathbb{G}$
 - $\text{Dec}'(k = g^r, c \in \mathbb{G})$: Output $m = k^{-1} \cdot c \in \mathbb{G}$

Correctness: $\text{Dec}'(k, \text{Enc}'(k, m)) = k \cdot m \cdot k^{-1} = m$

Security: Goal: $\forall m, m' \in \mathbb{G}, c \in \mathbb{G}, \Pr_{k \leftarrow \mathbb{G}} [\text{Enc}(k, m) = c] = \Pr_{k \leftarrow \mathbb{G}} [\text{Enc}(k, m') = c]$

Exercise: prove this (try to adapt proof from Lecture 1)

The Elgamal system (a concrete view)

- \mathbb{G} : finite cyclic group of prime order p with generator g
- $(\text{Enc}', \text{Dec}')$: symmetric-key encryption with keyspace $\mathcal{K} = \mathbb{G}$

Gen(1^n):

1. Sample $a \leftarrow \mathbb{Z}_p^*$
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The Elgamal system (a concrete view)

- \mathbb{G} : finite cyclic group of prime order p with generator g
- $(\text{Enc}', \text{Dec}')$: symmetric-key encryption with keyspace $\mathcal{K} = \mathbb{G}$

Gen(1^n):

1. Sample $a \leftarrow \mathbb{Z}_p^*$
2. Output (sk = a , pk = g^a)

Enc(pk, m):

1. Sample $b \leftarrow \mathbb{Z}_p^*$
2. Set $B = g^b$
3. Set $c := m \cdot \text{pk}^b = mg^{ab}$
4. Output $c' = (B, c)$

Dec(sk = a , (B, c)):

1. Compute $k = B^a$
2. Output $m = k^{-1}c$
 $= cg^{-ab}$
 $= mg^{ab}g^{-ab}$



What choice of $(\text{Enc}', \text{Dec}')$?

How to prove security?

Problem:
OTP uses random group element

But we only have g^{ab} !

Is this a problem? Isn't g^{ab} also random?

Problem: adversary *also* sees g^a and g^b !

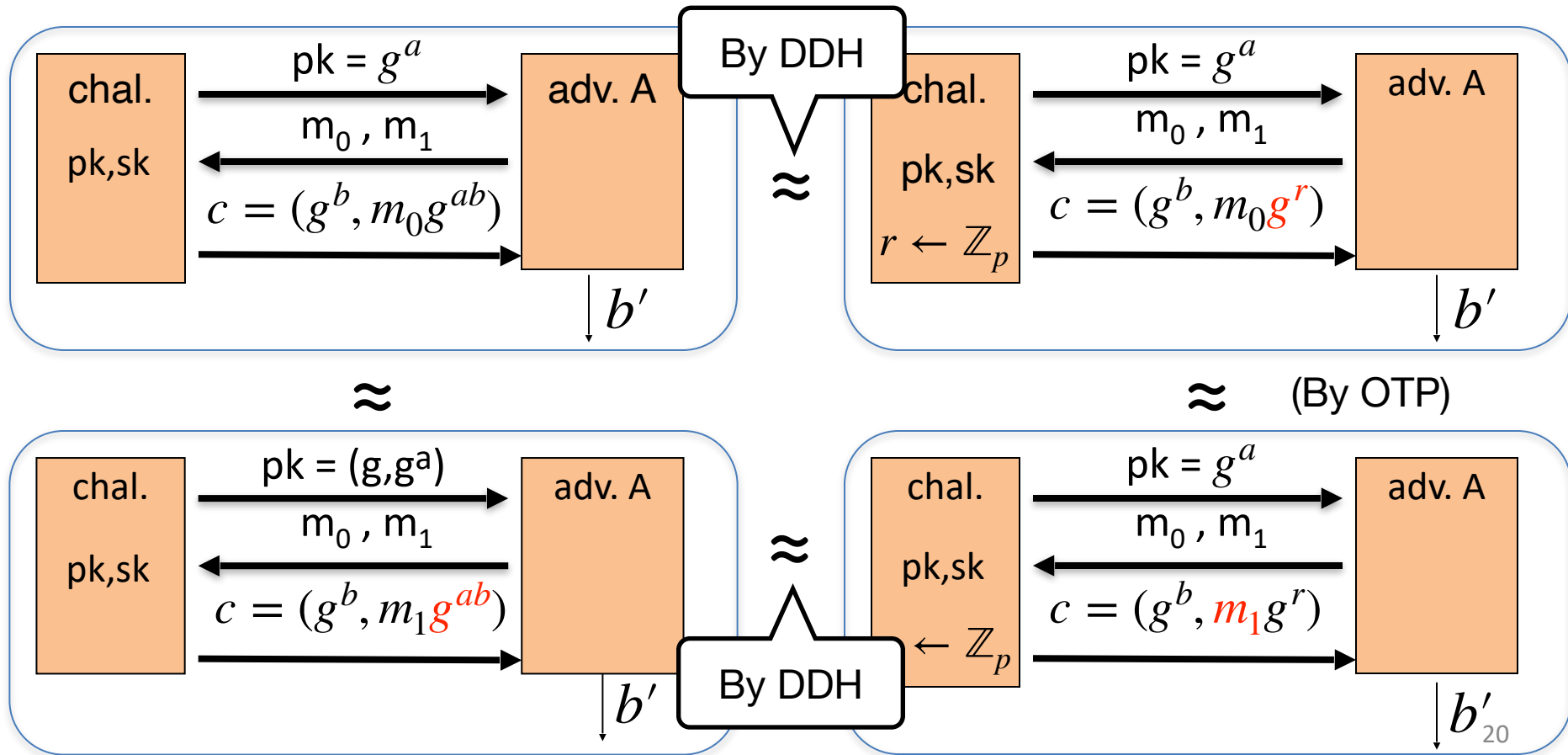
New assumption: Decisional Diffie–Hellman

Roughly, (g^a, g^b, g^{ab}) is indistinguishable from (g^a, g^b, g^r)

Formally, the following two distributions are computationally indistinguishable:

$$\{(g^a, g^b, g^{ab})\}_{a,b \leftarrow \mathbb{Z}_p} \text{ and } \{(g^a, g^b, g^r)\}_{a,b,r \leftarrow \mathbb{Z}_p}$$

Elgamal is semantically secure under DDH



The Elgamal system (a modern view)

- \mathbb{G} : finite cyclic group of prime order p with generator g
- $(\text{Enc}', \text{Dec}')$: what about arbitrary keyspace \mathcal{K} ?
- New ingredient: “Random”-ish hash function $H : \mathbb{G} \rightarrow \mathcal{K}$

Gen(1^n):

1. Sample $a \leftarrow \mathbb{Z}_p^*$
2. Output $(\text{sk} = a, \text{pk} = g^a)$

Enc(pk, m):

1. Sample $b \leftarrow \mathbb{Z}_p^*$
2. Set $k := H(g^{ab})$
3. Set $c \leftarrow \text{Enc}(k, m)$
4. Output $c' = (g^b, c)$

Dec(sk = a , (B, c)):

1. Compute $k = H(B^a)$
2. Output $m = \text{Dec}'(k, c)$

New assumption: Hash-DDH

Roughly, $(g^a, g^b, H(g^{ab}))$ is indistinguishable from (g^a, g^b, R)

Formally, the following two distributions are computationally indistinguishable:

$$\{(g^a, g^b, H(g^{ab}))\}_{a,b \leftarrow \mathbb{Z}_p} \text{ and } \{(g^a, g^b, R)\}_{a,b \leftarrow \mathbb{Z}_p, R \leftarrow \mathcal{K}}$$


Q: If DDH is hard, is H-DDH hard?

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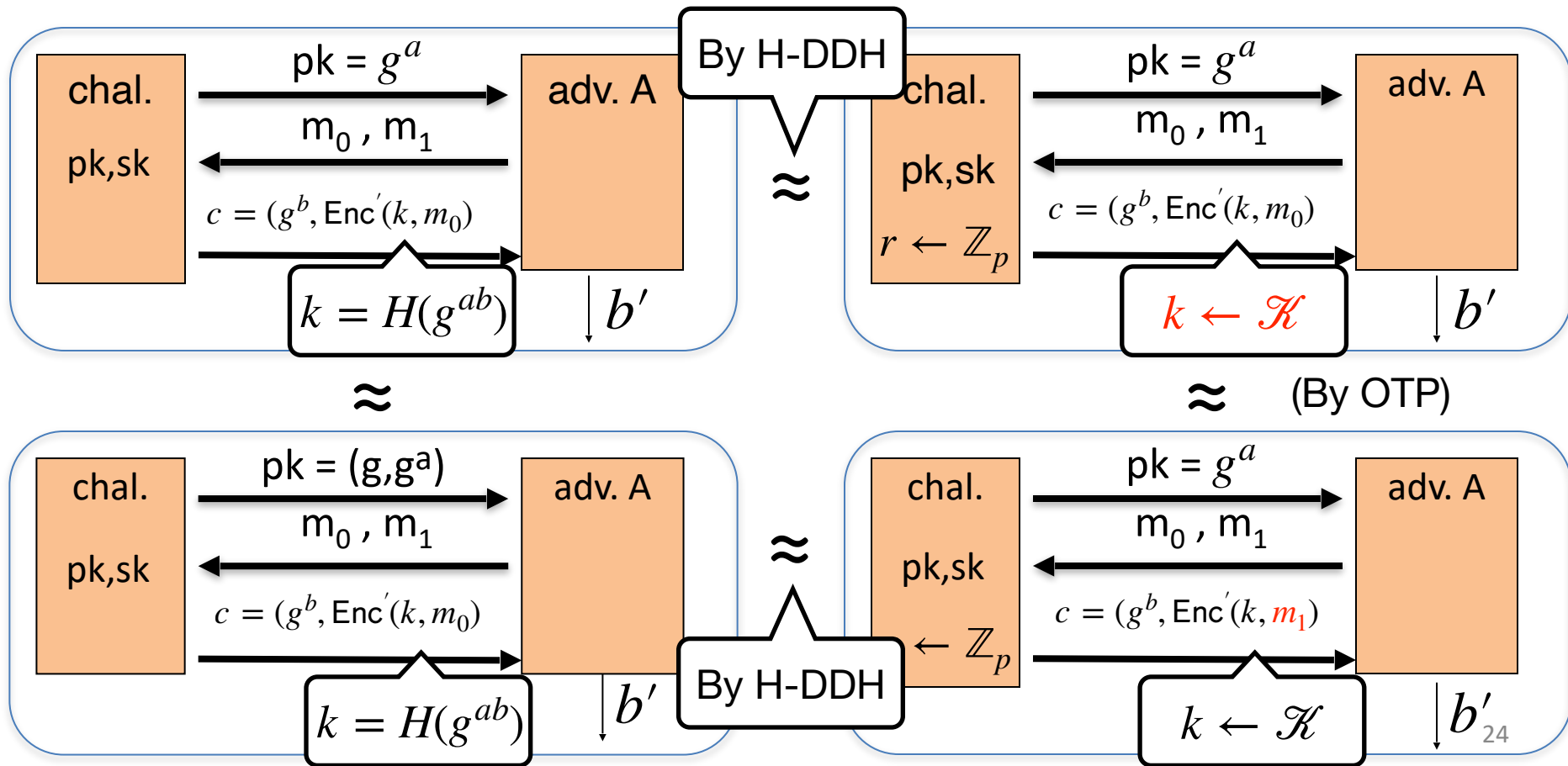
Suppose $K = \{0,1\}^{128}$ and

$H: G \rightarrow K$ only outputs strings in K that begin with 0
(i.e. for all y : $\text{msb}(H(y))=0$)

Can Hash-DH hold for (G, H) ?

- ☐ Yes, for some groups G
-  ☒ No, Hash-DH is easy to break in this case
- ☐ Yes, Hash-DH is always true for such H

Elgamal is semantically secure under H-DDH



Construction of PKE: Trapdoor Functions

Trapdoor functions (TDF)

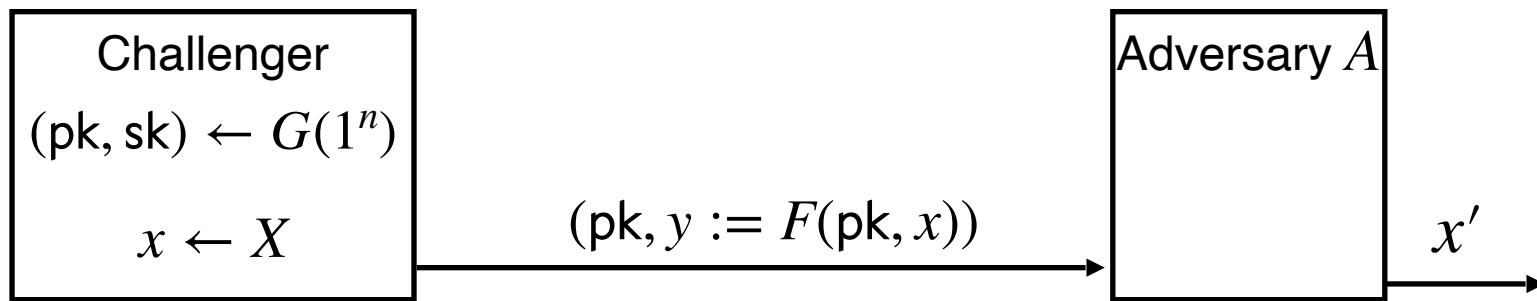
Def: A **trapdoor function** for input space X and output space Y is a triple of efficient algorithms (G, F, F^{-1})

- $G(1^n)$: randomized algorithm that outputs a key pair (pk, sk)
- $F(pk, \cdot)$: deterministic algorithm that computes $f : X \rightarrow Y$
- $F^{-1}(sk, \cdot)$: defines a function $Y \rightarrow X$ that inverts $F(pk, \cdot)$

More precisely: $\forall (pk, sk) \leftarrow G(1^n), \forall x \in X, F^{-1}(sk, F(pk, x)) = x$

Secure Trapdoor Functions (TDFs)

A TDF (G, F, F^{-1}) is secure if F_{pk} is a one-way function:



Def: (G, F, F^{-1}) is a **secure TDF** if for all efficient A :

$$\Pr \left[F(pk, x) = F(pk, x') \mid \begin{array}{l} (pk, sk) \leftarrow G(1^n) \\ x \leftarrow X \\ x' \leftarrow A(pk, F(pk, x)) \end{array} \right] = \text{negl}(n)$$

Construction: PKE from TDFs

PKE from Secure TDFs: Attempt 1

- (G, F, F^{-1}) : secure TDF $X \rightarrow Y$

Gen(1^n):

1. Output $(pk, sk) \leftarrow G(1^n)$.

Enc(pk, m):

1. Output $c \leftarrow F(pk, m)$.

Dec(sk, c):

1. Output $m := F^{-1}(sk, c)$.

Q: Is this secure?

A: No! Entirely deterministic \rightarrow cannot achieve IND-CPA!

PKE from Secure TDFs

- (G, F, F^{-1}) : secure TDF $X \rightarrow Y$
- $(\text{Gen}, \text{Enc}_s, \text{Dec}_s)$: symmetric AE defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$
- $H : X \rightarrow \mathcal{K}$: a hash function (like the one in Hashed ElGamal)

$\text{Gen}(1^n)$:

1. Output $(\text{pk}, \text{sk}) \leftarrow G(1^n)$.

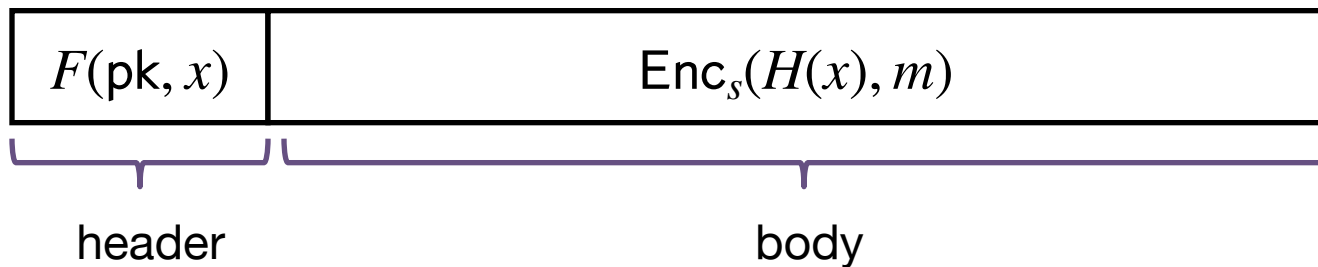
$\text{Enc}(\text{pk}, m)$:

1. Sample $x \leftarrow X$.
2. Compute key $k \leftarrow H(x)$
3. Output $(y \leftarrow F(\text{pk}, x), c \leftarrow \text{Enc}_s(k, m))$

$\text{Dec}(\text{sk}, (y, c))$:

1. Compute $x := F^{-1}(\text{sk}, y)$.
2. Compute key $k \leftarrow H(x)$
3. Output $\text{Dec}_s(k, c)$

In pictures:



Security Theorem:

If (G, F, F^{-1}) is a secure TDF,

(Gen, Enc_s, Dec_s) is an AE scheme, and

$H : X \rightarrow \mathcal{K}$ is a “random oracle”

then (Gen, Enc, Dec) is IND-CPA secure.

Review: Arithmetic modulo composites

Review: arithmetic mod composites

Let $N = pq$ where p, q are prime

$$\mathbb{Z}_N = \{0, 1, 2, \dots, N-1\}; \quad \mathbb{Z}_N^* = \{ \text{invertible elements in } \mathbb{Z}_N \}$$

Facts:

- $x \in \mathbb{Z}_N$ is invertible $\Leftrightarrow \gcd(x, N) = 1$
- Number of elements in \mathbb{Z}_N^* is $\varphi(N) = (p-1)(q-1) = N - p - q + 1$

Euler's thm: $\forall x \in \mathbb{Z}_N^* : x^{\varphi(N)} = 1$

Modular e -th roots

We know how to solve modular **linear** equations:

$$ax + b = 0 \text{ in } \mathbb{Z}_N$$

$$\text{Solution: } x = -b \cdot a^{-1} \text{ in } \mathbb{Z}_N$$

(inverses are fast even for N composite)

What about higher degree polynomials?

Example: Let $N = pq$ for two primes p, q .

Given an arbitrary $y \in \mathbb{Z}_N$, can we find x such that

$$y = x^e \pmod{N}?$$

Answering these questions requires the factorization of N
(as far as we know)

The factoring problem

Gauss (1805): *“The problem of distinguishing prime numbers from composite numbers and of resolving the latter into their prime factors is known to be one of the most important and useful in arithmetic.”*

Best known alg. (NFS): run time $2^{O(\sqrt[3]{n})}$ for n -bit integer

Current world record: **RSA-768** (232 digits)

- Work: two years on hundreds of machines
- Factoring a 1024-bit integer: about 1000 times harder
⇒ likely possible this decade

Key lengths

Security of public key system should be comparable to security of symmetric cipher:

Cipher key-size

80 bits

128 bits

256 bits (AES)

RSA

Modulus size

1024 bits

3072 bits

15360 bits

Construction of Trapdoor Functions

Big question:
can we use hardness of computing
 e -th roots to construct a secure TDF?

Secure TDFs from e -th roots

Gen(1^n):

1. Sample primes $p, q \sim 1024$ bits
2. Set $N = pq$
3. Sample e, d s.t. $e = d^{-1} \pmod{\varphi(N)}$
4. Set $\text{sk} = (p, q, d)$ and $\text{pk} := (N, e)$
5. Output (pk, sk) .

$F(\text{pk} = (N, e), x)$:

1. Output $x^e \pmod N$.

Dec($\text{sk} = (p, q, d), y$):

1. Output $x := y^d \pmod N$.

Correctness: $\forall (\text{pk}, \text{sk}) \leftarrow G(1^n), \forall x \in X, F^{-1}(\text{sk}, F(\text{pk}, x)) = x ?$

$$F_{\text{sk}}^{-1}(F_{\text{pk}}(x)) = (x^e)^d \equiv x^{ed} \pmod{\varphi(N)} \equiv x^{1+k\varphi(N)} \equiv x \pmod N$$

This is called the RSA Trapdoor permutation

First published: Scientific American, Aug. 1977.

Very widely used:

- SSL/TLS: certificates and key-exchange
- Secure e-mail and file systems
- ... many others

Secure TDFs from e -th roots

Gen(1^n):

1. Sample primes $p, q \sim 1024$ bits
2. Set $N = pq$
3. Sample e, d s.t. $e = d^{-1} \pmod{\varphi(N)}$
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5. Output (pk, sk) .

$F(\text{pk} = (N, e), x)$:

1. Output $x^e \pmod N$.

Dec($\text{sk} = (p, q, d), y$):

1. Output $x := y^d \pmod N$.

Security?

By “assumption”

RSA assumption: Roughly, computing e -th roots is hard

$$\Pr \left[A(\text{pk}, x^e \bmod N) = x \mid (\text{pk} = (N, d), \text{sk} = (p, q, e)) \leftarrow G(1^n), x \leftarrow X \right] = \text{negl}(n)$$

The RSA TDF is actually a
trapdoor *permutation*

Is the RSA assumption plausible?

To invert the RSA one-way func. (without d) attacker must compute

$$x \text{ from } c = x^e \bmod N.$$

How hard is computing e -th roots modulo N ??

Best known algorithm:

- Step 1: factor N (hard)
- Step 2: compute e -th roots modulo p and q (easy)

Shortcuts?

Must one factor N in order to compute e -th roots?

To prove no shortcut exists we need a reduction:

- Efficient algorithm for e -th roots mod N
 \Rightarrow efficient algorithm for factoring N .
- Oldest problem in public key cryptography.

Some evidence no reduction exists: (BV'98)

- “Algebraic” reduction \Rightarrow factoring is easy.

Textbook RSA is insecure

Textbook RSA encryption:

- public key: (N, e) Encrypt: $c \leftarrow m^e \quad (\text{in } Z_N)$
- secret key: (N, d) Decrypt: $c^d \rightarrow m$

Insecure cryptosystem !!

- Is not semantically secure and many attacks exist

⇒ The RSA trapdoor permutation is not an encryption scheme !

RSA in practice

How **not** to improve RSA's performance

To speed up RSA decryption use small private key d ($d \approx 2^{128}$)

$$c^d = m \pmod{N}$$

Wiener'87: if $d < N^{0.25}$ then RSA is insecure.

BD'98: if $d < N^{0.292}$ then RSA is insecure (open:
 $d < N^{0.5}$)

Insecure: priv. key d can be found from (N, e)

RSA With Low public exponent

To speed up RSA encryption use a small e : $c = m^e \pmod{N}$

- Minimum value: $e=3$ ($\gcd(e, \varphi(N)) = 1$)
- Recommended value: $e=65537=2^{16}+1$

Encryption: 17 multiplications

Asymmetry of RSA: fast enc. / slow dec.

- ElGamal (next module): approx. same time for both.